MAXIMUM RANGE FOR A ROCKET IN HORIZONTAL FLIGHT*

(O MAKSIMAL'NOI DLINE POLETA RAKETY V GORIZONTAL'NOI PLOSKOSTI)

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This note deals with the derivation of the optimal thrust control of a rocket vehicle constrained to fly in the horizontal plane in order to attain maximum range. It is assumed that the gravitational acceleration is constant and the aerodynamic drag is approximated by the so-called parabolic drag polar.** It is shown that the optimal thrust program consists of at most three thrust regimes. The uniqueness, and hence the global nature of the maximum, is shown.

The equations of motion are

$$\dot{V} + \frac{D}{m} + \frac{c\dot{m}}{m} = 0, \qquad \dot{x} - V = 0 \tag{1}$$

$$t = 0, \quad x = 0, \quad m = m_1, \quad V = V_1; \qquad t = t_2, \quad m = m_2, \quad V = V_2, \quad x(t_2) = \max$$

where g is the acceleration of gravity (constant); m is mass; x is range; V is speed; D is drag; c is exhaust speed (constant) and ($^\cdot$) represents d/dt.

Here we shall assume a parabolic drag polar

$$D = AV^2 + BL^2 \qquad A = \text{const}, \quad B = \text{const}) \tag{2}$$

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* This is the original English text of this American paper, of which the Russian version appeared in PMM Vol. 27, No. 3, 1963.

** The same problem has been treated earlier, [1], for a drag of the form $D = AV^2 + BL^2/V^2$, A and B constants. The derivation in [1] is quite long and involved, however. The "singular" portion of the solution was also found in [2].

where lift

$$L = mg \tag{3}$$

We shall also take the thrust $T = -c\dot{m}$ to be bounded

$$0 \leqslant -c\dot{m} \leqslant T_{\max} \tag{4}$$

and assume that T_{max} sufficiently large so that

$$T_{\max} \geqslant D_{\max} \tag{5}$$

We now introduce nondimensional variables

$$\tau = \frac{g}{c}t, \qquad \xi = \frac{g}{c^2}x, \qquad \mu = \frac{m}{m_1}, \qquad v = \frac{V}{c} \tag{6}$$

so that the equations of motion become

$$v' + \frac{D(\mu, v)}{\mu} - \frac{\beta}{\mu} = 0, \qquad \xi' - v = 0, \qquad \mu' + \beta = 0 \tag{7}$$

$$\tau = 0, \quad \xi = 0, \quad \mu = 1, \quad v = v_1 \qquad \tau = \tau_2, \quad \mu = \mu_2, \quad v = v_2, \quad \xi(\tau_2) = \max$$

where

$$D = \frac{Ac^2}{m_1 g}v^2 + Bm_1 g\mu^2 = av^2 + b\mu^2 \tag{8}$$

Primes indicate differentiation with respect to dimensionless time.

Inequality constraint (4) we shall state as

$$0 \leqslant \beta \leqslant \beta_{\max} \tag{9}$$

The problem is to determine the optimal thrust $\beta(\tau)$ so that the range $\xi(\tau_2)$ is maximum, i.e.

$$G = -\xi(\tau_2) \tag{10}$$

is minimum [3,4].

1. Adjoint equations:

$$\lambda_v' - \lambda_v \frac{D_v}{\mu} + \lambda_\xi = 0, \qquad \lambda_\xi' = 0, \qquad \lambda_\mu' + \lambda_v\left(\frac{D-\beta}{\mu^2} - \frac{D_\mu}{\mu}\right) = 0 \tag{11}$$

2. Optimal control:

$$\max_\beta H, \qquad H = \lambda_v v' + \lambda_\xi \xi' + \lambda_\mu \mu' = \beta\left(\frac{\lambda_v}{\mu} - \lambda_\mu\right) - \lambda_v \frac{D}{\mu} + \lambda_\xi v \tag{12}$$

i.e. H is linear in control β and switching function

$$K \equiv \frac{\lambda_v}{\mu} - \lambda_\mu \tag{13}$$

so that

$$\beta = \beta_{\max} \quad \text{if} \quad K > 0, \qquad \beta = 0 \quad \text{if} \quad K < 0, \qquad \beta = \beta(\tau) \quad \text{for} \quad K \equiv 0 \tag{14}$$

The "singular" arc $K \equiv 0$ may be admissible if the corresponding $\beta(\tau)$ is admissible, i.e. satisfies (9).

3. Transversality condition:

$$-d\xi(\tau_2) + [\lambda_v\, dv + \lambda_\xi d\xi + \lambda_\mu d_\mu - H\, d\tau]_1^2 = 0 \tag{15}$$

In view of the end conditions

$$d\xi(0) = dv(0) = dv(\tau_2) = d\mu(0) = d\mu(\tau_2) = d\tau_1 = 0 \tag{16}$$

so that

$$\lambda_\xi(\tau_2) = 1, \qquad H(\tau_2) = 0 \tag{17}$$

4. First integral:

Since equations (7) are autonomous and H is continuous

$$H \equiv 0 \tag{18}$$

is a first integral. It can replace any one of the adjoint equations (11).

5. Corner conditions:

H and the λ_i are continuous.

6. Synthesis of optimal control:

First we shall investigate the "singular" solution $K \equiv 0$. For this solution

$$K' = 0 \tag{19}$$

as well. Now upon differentiation of K and use of (18) with $K = 0$, namely

$$-\lambda_v \frac{D}{\mu} + \lambda_\xi v = 0 \tag{20}$$

together with the first equation in (17) and (11), we get

$$K' = \frac{v}{\mu} - \frac{1}{\mu} + \frac{v}{\mu}\frac{D_v}{D} - \frac{D_\mu}{D} \tag{21}$$

i.e. for drag of form (8)

$$K' = \frac{D}{\mu}(v+1)(av^2 - b\mu^2) \tag{22}$$

so that, for $v > 0$, (19) becomes

$$\mu = \sqrt{a/b}\; v \tag{23}$$

Figure 1 shows this solution.

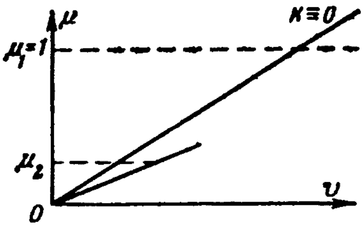

Fig. 1.

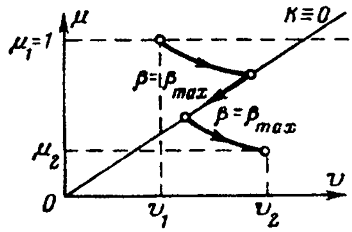

Fig. 2.

With assumption (5), i.e.

$$\beta_{\max} \geqslant D_{\max} \tag{24}$$

the singular solution certainly satisfies (9), since

$$\mu' = -\beta = \sqrt{\frac{a}{b}}\, v' = \sqrt{\frac{a}{b}}\,\frac{\beta - D}{\mu} \tag{25}$$

i.e.

$$0 < \beta = \left(\mu + \sqrt{\frac{a}{b}}\right)^{-1} D \leqslant \beta_{\max} \tag{26}$$

Now end conditions may be such that initial and end points may lie either to the left or the right of the "singular" arc $K \equiv 0$. Figure 2 shows one of these cases together with "boundary arcs" connecting the "singular" arc to the end points.

We suspect that this is the solution. However, we must show that it is the only admissible one. Figure 3 shows other possibilities.

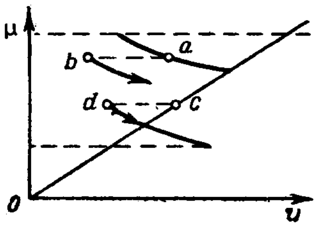

Fig. 3.

It is easy to show that such arcs are not admissible. For, at corners

$$K = 0 \tag{27}$$

since it is continuous. Hence, at corner

$$\begin{array}{lll} a) & K' < 0, & \text{i.e. } av^2 - b\mu^2 < 0 \\ b) & K' > 0, & \text{i.e. } av^2 - b\mu^2 > 0 \end{array} \tag{28}$$

but

$$\mu(a) = \mu(b), \qquad v(a) > v(b) \tag{29}$$

which contradicts (28). A similar argument rules out corners c and d, as well as all such other corners.*

Finally, Fig. 4 shows the solution for other end conditions.

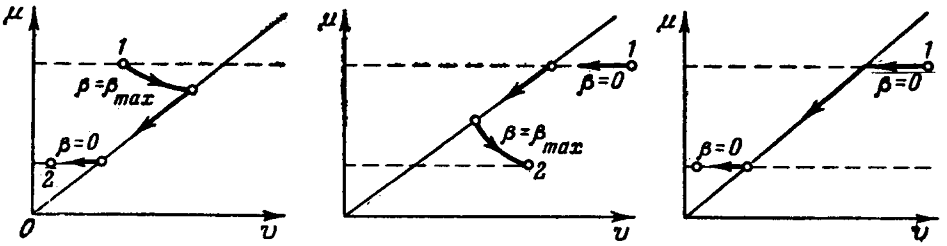

Fig. 4.

In conclusion we note:

a) The solution is made up of at most three thrust regimes.

b) Since the solution for the local maximum is unique, it yields a global maximum (as borne out by the application of the Green's Theorem Method, Chapter 3 of *Optimization Techniques*, [5]).

* Note that the occurrence of more than one corner in either region is ruled out, since at a corner, $K = 0$, $K' < 0$ above and $K' > 0$ below the singular solution line.

BIBLIOGRAPHY

1. Hibbs, A.R., "Optimum Burning Program for Horizontal Flight," *J. Amer. Rocket Soc.*, 22, p. 204, 1952.

2. Ross, S., "Minimality for Problems in Vertical and Horizontal Rocket Flight," *Jet Prop.*, 28, p. 55, 1958.

3. Pontriagin, L.S., Boltianskii, V.G., Gamkrelidze, R.V. and Mishchenko, E.F., *The Mathematical Theory of Optimal Processes*, Interscience Publishers, 1962.

4. Leitmann, G., "An Elementary Derivation of the Optimal Control Conditions," *12th Int. Astronaut. Cong.*, Washington, 1961.

5. Miele, A., Chapter 3 of *Optimization Techniques*, ed. G. Leitmann, Academic Press, 1962.